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Star democracy in open string field theory

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Abstract: We study three types of star products in SFT: the ghosts, the twisted ghosts and the matter. We find that their Neumann coefficients are related to each other in a compact way which includes the Gross-Jevicki relation between matter and ghost sector: we explicitly show that the same relation, with a minus sign, holds for the twisted and nontwisted ghosts (which are different but define the same solution). In agreement with this, we prove that matter and twisted ghost coefficients just differ by a minus sign. As a consistency check, we also compute the spectrum of the twisted ghost vertices from conformal field theory and, using equality of twisted and reduced slivers, we derive the spectrum of the non twisted ghost star.

Keywords: Bosonic Strings, String Field Theory.

Contents

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1. Introduction

Open Bosonic String Theory is made up with two sectors, the embedding coordinates in the 26 dimensional target space (which we call the matter), and a bc system with conformal weights $(2, -1)$ (which we call the ghosts).

The second quantized version is known to be Witten String Field Theory [1]. In this theory the interaction between string fields is encoded in the star product which defines an associative non commutative multiplication in the algebra of string fields.

This theory has lead to a successful study of the process of tachyon condensation [2]–[5], and hence to the identification of a stable vacuum of open string theory.

The theory which is supposed to describe strings around the stable vacuum were postulated in [6], under the name of "Vacuum String Field Theory"(VSFT). Such theory is formally identical to Witten's one, except for the kinetic operator, which is no more the usual BRST charge, but a c-midpoint insertion. In such a theory the ghost sector is completely decoupled from the matter, hence solutions can be found in a factorized form *matter* $\otimes$ *ghost*. Such solutions were obtained following two parallel methods: one which is algebraic and is based on the oscillator expansion of the string field [7]–[9], the other is based on the Boundary Conformal Theory which describes the original unstable D-brane configuration [10, 11].

The two approaches, although very different in the formalism, were shown to lead to the same results, first numerically by level expansion analysis, and then analytically in [12, 13] by making use of the continuous basis of the star product.

Even if the ghost sector of the theory is decoupled from the matter, nevertheless it is crucial for the consistency of VSFT, since it is supposed to contain all the tree level dynamics of open strings at the tachyon vacuum. For this reason a careful study of this sector is needed in order to understand better what is the physics behind it. One of the aims of this paper is indeed to shed some light on the ghost star product and its different versions (reduced and twisted) which have been used in the algebraic and conformal methods.

Moreover, if for the matter the correspondence between algebraic and conformal approach is quite clear and actually relies on the isomorphism between CFT fields and their Fourier transformed oscillators, the correspondence in the ghost sector is more subtle since it compares, on the one hand, projectors squeezed states in Siegel gauge build up with oscillators of the bc system [8] and, on the other hand, projectors surface states in a bc CFT, twisted by one unit of ghost current [10]; in this way the star product is ghost number preserving and we can properly define projectors.

Since, when restricted to Siegel gauge, we can define the reduced star product which is also ghost number preserving, it was then natural to identify the reduced star product on the algebraic side with the twisted star product on the BCFT side. This is actually not correct,[1] as we are going to show, since the two product are different (they have different Neumann coefficients) but, curiously enough, they define the same sliver-projectors.

The picture which arise at the end of our analysis is that of three star products (matter, ghost and twisted ghost); each of them can be defined completely from the underlying CFT and all sliver-like projectors have the same (up to a minus sign) Neumann coefficients. It is interesting to note that such equality of solutions is implied in a bijective way by the Gross-Jevicki relation [14] which connects the ghost Neumann coefficients to the matter ones. Moreover the relevant structures of the matter star product (at least at zero momentum) are completely encoded in the Neumann coefficients of the twisted bc system. This fact puts the bc CFT (with its twisted variant) in an equivalent position with respect to the matter.

The paper is organized as follows. In section 2 we briefly recall definitions and properties of the matter and ghost product from a conformal point of view, as done in [15],[2] then we perform the twist as in [10] and define properly the twisted star product, which turns out to be very closely related to the matter product.

In section 3 we determine the relations that connect the three vertices in the game. In particular we show that the twisted CFT defines, up to a sign, the same coefficients as the matter.

[1] We thank L. Rastelli for pointing out this.

[2] This representation of the interaction vertex is equivalent to the operatorial construction based on delta overlaps [16, 14, 17].

In section 4, after a brief review of matter and ghost algebraic projectors, we derive the algebraic expression of the twisted sliver and identify it with the sliver-like state in Siegel gauge, through the Gross-Jevicki relation [14].

Section 5 is devoted to spectral analysis of the 2 star products in the ghost sector: one of them can be naturally casted in a matrix structure which exhibits a discrete spectrum consisting of one eigenvalue and one eigenvector (a c midpoint insertion) plus a continuous spectrum. Here the use of the twisted CFT allows us to derive the continuous spectrum of the twisted product following the same conformal methods of [18] and equality of twisted and non twisted slivers leads to the spectrum of the reduced product.[3]

2. The three stars

In this section we briefly review the construction of the interaction vertex of matter and ghost sector. Details will be almost skipped and can be found in [15]

2.1 Matter star

The matter part of the three strings vertex [1, 16, 14] is given by

$$|V_3\rangle = \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\delta^{26}(p_{(1)}+p_{(2)}+p_{(3)})\exp(-E)\,|0,p\rangle_{123} \tag{2.1}$$

where

$$E = \sum_{a,b=1}^{3}\left(\frac{1}{2}\sum_{m,n\geq 1}\eta_{\mu\nu}a_m^{(a)\mu\dagger}V_{mn}^{ab}a_n^{(b)\nu\dagger} + \sum_{n\geq 1}\eta_{\mu\nu}p_{(a)}^{\mu}V_{0n}^{ab}a_n^{(b)\nu\dagger} + \frac{1}{2}\eta_{\mu\nu}p_{(a)}^{\mu}V_{00}^{ab}p_{(b)}^{\nu}\right). \tag{2.2}$$

Summation over the Lorentz indices $\mu,\nu = 0,\ldots,25$ is understood and η denotes the flat Lorentz metric. The operators $a_m^{(a)\mu}, a_m^{(a)\mu\dagger}$ denote the non-zero modes matter oscillators of the a-th string, which satisfy

$$[a_m^{(a)\mu}, a_n^{(b)\nu\dagger}] = \eta^{\mu\nu}\delta_{mn}\delta^{ab}\,, \qquad m,n\geq 1 \tag{2.3}$$

$p_{(r)}$ is the momentum of the a-th string and $|0,p\rangle_{123} \equiv |p_{(1)}\rangle\otimes|p_{(2)}\rangle\otimes|p_{(3)}\rangle$ is the tensor product of the Fock vacuum states relative to the three strings. $|p_{(a)}\rangle$ is annihilated by the annihilation operators $a_m^{(a)\mu}$ and it is eigenstate of the momentum operator $\hat{p}_{(a)}^{\mu}$ with eigenvalue $p_{(a)}^{\mu}$. The normalization is

$$\langle p_{(a)}|\,p'_{(b)}\rangle = \delta_{ab}\delta^{26}(p+p')\,.$$

The conformal definition of the vertex starts with the gluing functions

$$f_a(z_a) = \alpha^{2-a}f(z_a)\,, \quad a = 1,2,3 \tag{2.4}$$

[3]On the same day the paper [19] appeared, with some overlapping with our results.

where

$$f(z) = \left(\frac{1+iz}{1-iz}\right)^{\frac{2}{3}}$$
$$\alpha = e^{\frac{2\pi i}{3}}\,. \tag{2.5}$$

The interaction vertex is defined by a correlation function on the disk in the following way

$$\int \psi * \phi * \chi = \langle f_1 \circ \psi(0)\, f_2 \circ \phi(0)\, f_3 \circ \chi(0)\rangle = \langle V_3|\psi\rangle_1|\phi\rangle_2|\chi\rangle_3\,. \tag{2.6}$$

Now we consider the string propagator at two generic points of this disk. The Neumann coefficients N^{ab}_{nm} are nothing but the Fourier modes of the propagator with respect to the original coordinates z_a. We shall see that such Neumann coefficients are related in a simple way to the standard three strings vertex coefficients. Here we will deal only with the zero momentum vertex, which is the one which is strictly connected to the (twisted) ghost vertex.

The Neumann coefficients N^{ab}_{mn} are given by [20]

$$N^{ab}_{mn} = \langle V_3|\alpha^{(a)}_{-n}\alpha^{(b)}_{-m}|0\rangle_{123} = -\frac{1}{nm}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^m}f'_a(z)\frac{1}{(f_a(z)-f_b(w))^2}f'_b(w) \tag{2.7}$$

where the contour integrals are understood around the origin. It is easy to check that

$$N^{ab}_{mn} = N^{ba}_{nm}$$
$$N^{ab}_{mn} = (-1)^{n+m}N^{ba}_{mn}$$
$$N^{ab}_{mn} = N^{a+1,b+1}_{mn}\,. \tag{2.8}$$

In the last equation the upper indices are defined mod 3.

Let us consider the decomposition

$$N^{ab}_{nm} = \frac{1}{3\sqrt{nm}}\Big(C_{nm} + \bar\alpha^{a-b}U_{nm} + \alpha^{a-b}\bar U_{nm}\Big)\,. \tag{2.9}$$

After some algebra one gets

$$C_{nm} = \frac{-1}{\sqrt{nm}}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^m}\left(\frac{1}{(1+zw)^2}+\frac{1}{(z-w)^2}\right)$$
$$U_{nm} = \frac{-1}{3\sqrt{nm}}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^m}\left(\frac{f^2(w)}{f^2(z)}+2\frac{f(z)}{f(w)}\right)\left(\frac{1}{(1+zw)^2}+\frac{1}{(z-w)^2}\right)$$
$$\bar U_{nm} = \frac{-1}{3\sqrt{nm}}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^m}\left(\frac{f^2(z)}{f^2(w)}+2\frac{f(w)}{f(z)}\right)\left(\frac{1}{(1+zw)^2}+\frac{1}{(z-w)^2}\right)\,. \tag{2.10}$$

The integrals can be directly computed in terms of the Taylor coefficients of f. The result is

$$C_{nm} = (-1)^n\delta_{nm} \tag{2.11}$$
$$U_{nm} = \frac{1}{3\sqrt{nm}}\sum_{l=1}^{m} l\Big[(-1)^n B_{n-l}B_{m-l} + 2b_{n-l}b_{m-l}(-1)^m - $$
$$- (-1)^{n+l}B_{n+l}B_{m-l} - 2b_{n+l}b_{m-l}(-1)^{m+l}\Big] \tag{2.12}$$
$$\bar U_{nm} = (-1)^{n+m}U_{nm} \tag{2.13}$$

where we have set

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} b_k z^k \\ f^2(z) &= \sum_{k=0}^{\infty} B_k z^k \,, \quad \text{i.e.} \quad B_k = \sum_{p=0}^{k} b_p b_{k-p} \,. \end{aligned} \tag{2.14}$$

Using the integral representation (2.10) one can prove [15]

$$\sum_{k=1}^{\infty} U_{nk} U_{km} = \delta_{nm} \,, \qquad \sum_{k=1}^{\infty} \bar{U}_{nk} \bar{U}_{km} = \delta_{nm} \,. \tag{2.15}$$

In order to make contact with the standard notations (for example [6]) we define[4]

$$V^{ab}_{nm} = (-1)^{n+m} \sqrt{nm}\, N^{ab}_{nm} \tag{2.16}$$

and

$$\begin{aligned} M &= CV^{11} \\ M_+ &= CV^{12} \\ M_- &= CV^{21} \,. \end{aligned} \tag{2.17}$$

Using (2.15), together with the decomposition (2.9), it is easy to establish the following linear and non linear relations (written in matrix notation).

$$\begin{aligned} M + M_+ + M_- &= 1 \\ M^2 + M_+^2 + M_-^2 &= 1 \\ M_+^3 + M_-^3 &= 2M^3 - 3M^2 + 1 \\ M_+ M_- &= M^2 - M \\ [M, M_\pm] &= 0 \\ [M_+, M_-] &= 0 \,. \end{aligned} \tag{2.18}$$

2.2 Ghost star

To start with we define, in the ghost sector, the vacuum states $|\hat{0}\rangle$ and $|\dot{0}\rangle$ as follows

$$|\hat{0}\rangle = c_0 c_1 |0\rangle \,, \qquad |\dot{0}\rangle = c_1 |0\rangle \tag{2.19}$$

where $|0\rangle$ is the usual $\mathrm{SL}(2, \mathbb{R})$ invariant vacuum. Using *bpz* conjugation

$$c_n \to (-1)^{n+1} c_{-n} \,, \qquad b_n \to (-1)^{n-2} b_{-n} \,, \qquad |0\rangle \to \langle 0| \tag{2.20}$$

one can define conjugate states.

[4] The factor $(-1)^{n+m}$ is there because these coefficients refer to the Ket vertex $|V_3\rangle$, so *bpz* is needed.

The three strings interaction vertex is defined, as usual, as a squeezed operator acting on three copies of the bc Hilbert space

$$\langle\tilde{V}_3| = {}_1\langle\hat{0}|\,{}_2\langle\hat{0}|\,{}_3\langle\hat{0}|e^{\tilde{E}}\,, \qquad \tilde{E} = \sum_{a,b=1}^{3}\sum_{n,m}^{\infty} c_n^{(a)}\tilde{N}_{nm}^{ab}b_m^{(b)}\,. \tag{2.21}$$

Under bpz conjugation

$$|\tilde{V}_3\rangle = e^{\tilde{E}'}|\hat{0}\rangle_1|\hat{0}\rangle_2|\hat{0}\rangle_3\,, \qquad \tilde{E}' = -\sum_{a,b=1}^{3}\sum_{n,m}^{\infty}(-1)^{n+m}c_n^{(a)\,\dagger}\tilde{N}_{nm}^{ab}b_m^{(b)\,\dagger}\,. \tag{2.22}$$

To make the propagator $\mathrm{SL}(2,\mathbb{R})$ we have to insert three c zero modes at points ξ_i [20]

$$\langle b(z)c(w)\rangle = \frac{1}{z-w}\prod_{i=1}^{3}\frac{w-\zeta_i}{z-\zeta_i}\,. \tag{2.23}$$

So we get

$$\begin{aligned}\tilde{N}_{nm}^{ab} &= \langle\tilde{V}_3|b_{-n}^{(a)}c_{-m}^{(b)}|\hat{0}\rangle_{123}\\ &= \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n-1}}\frac{1}{w^{m+2}}(f_a'(z))^2\frac{-1}{f_a(z)-f_b(w)}\prod_{i=1}^{3}\frac{f_b(w)-f_i(0)}{f_a(z)-f_i(0)}(f_b'(w))^{-1}\,.\end{aligned} \tag{2.24}$$

It is straightforward to check that

$$\tilde{N}_{nm}^{ab} = \tilde{N}_{nm}^{a+1,b+1} \tag{2.25}$$

and (by letting $z\to -z$, $w\to -w$)

$$\tilde{N}_{nm}^{ab} = (-1)^{n+m}\tilde{N}_{nm}^{ba}\,. \tag{2.26}$$

As in the matter case, we consider the decomposition

$$\tilde{N}_{nm}^{ab} = \frac{1}{3}(\tilde{C}_{nm} + \bar{\alpha}^{a-b}\tilde{U}_{nm} + \alpha^{a-b}\bar{\tilde{U}}_{nm})\,. \tag{2.27}$$

After some algebra one finds

$$\begin{aligned}\tilde{C}_{nm} &= \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right)\\ \tilde{U}_{nm} &= \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\frac{f(z)}{f(w)}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right)\\ \bar{\tilde{U}}_{nm} &= \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\frac{f(w)}{f(z)}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right)\end{aligned} \tag{2.28}$$

It is easy to show that

$$\bar{\tilde{U}}_{nm} = (-1)^{n+m}\tilde{U}_{nm}\,. \tag{2.29}$$

As discussed in detail in [15] the evaluation of these integrals is sensible to radial ordering in the $(n,-n)$, components. We fix the ambiguity by setting

$$\tilde{N}_{-1,1}^{aa} = \tilde{N}_{1,-1}^{aa} = 0\,, \qquad \tilde{N}_{0,0}^{aa} = 1\,. \tag{2.30}$$

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Which corresponds to

$$\tilde{C}_{NM} = (-1)^N \delta_{NM} \quad N, M \geq 0 \tag{2.31}$$

$$\tilde{U}_{NM} = (-1)^M b_N b_M + (-1)^M \sum_{l=1}^{M} (b_{N-l} b_{M-l} + (-1)^l b_{N+l} b_{M-l}) \tag{2.32}$$

where the b_n's have been defined in (2.14). The reason for this is that we get the fundamental identity

$$\sum_{K=0} \tilde{U}_{NK} \tilde{U}_{KM} = \delta_{NM} \,. \tag{2.33}$$

As for the matter case we will consider from now on the coefficients of the Ket vertex

$$\mathcal{V}^{ab}_{NM} = -(-1)^{n+m} \tilde{N}^{ab}_{NM} \,. \tag{2.34}$$

Then we define

$$\begin{aligned} \mathcal{Y} &= C\mathcal{V}^{11} \\ \mathcal{Y}_+ &= C\mathcal{V}^{12} \\ \mathcal{Y}_- &= C\mathcal{V}^{21} \end{aligned}$$

From (2.33) we obtain the linear and non linear relations

$$\begin{aligned} \mathcal{Y} + \mathcal{Y}_+ + \mathcal{Y}_- &= 1 \\ \mathcal{Y}^2 + \mathcal{Y}_+^2 + \mathcal{Y}_-^2 &= 1 \\ \mathcal{Y}_+^3 + \mathcal{Y}_-^3 &= 2\mathcal{Y}^3 - 3\mathcal{Y}^2 + 1 \\ \mathcal{Y}_+ \mathcal{Y}_- &= \mathcal{Y}^2 - \mathcal{Y} \\ [\mathcal{Y}, \mathcal{Y}_\pm] &= 0 \\ [\mathcal{Y}_+, \mathcal{Y}_-] &= 0 \,. \end{aligned} \tag{2.35}$$

Note that the matrices $\mathcal{Y}$ have the structure

$$\mathcal{Y} = \begin{pmatrix} 1 & 0 \\ \vec{y} & Y \end{pmatrix} \tag{2.36}$$

$$\mathcal{Y}_\pm = \begin{pmatrix} 0 & 0 \\ \vec{y}_\pm & Y_\pm \end{pmatrix}. \tag{2.37}$$

The "small" matrices $Y, Y_\pm$ are the coefficients of the reduced star product $*_{b_0}$, defined as

$$A *_{b_0} B = b_0 (A * B) \,. \tag{2.38}$$

From (2.36) the following properties of "small" matrices and column vectors relative to the b_0 modes (hence excluded from the reduced product) are inherited

$$\begin{aligned} Y + Y_+ + Y_- &= 1 \\ \vec{y} + \vec{y}_+ + \vec{y}_- &= 0 \end{aligned}$$

$$
\begin{aligned}
Y^2 + Y_+^2 + Y_-^2 &= 1 \\
(1+Y)\vec{y} + Y_+\vec{y}_+ + Y_-\vec{y}_- &= 0 \\
Y_+^3 + Y_-^3 &= 2Y^3 - 3Y^2 + 1 \\
Y_+^2\vec{y}_+ + Y_-^2\vec{y}_- &= (2Y^2 - Y - 1)\vec{y} \\
Y_+Y_- &= Y^2 - Y \\
[Y, Y_\pm] &= 0 \\
[Y_+, Y_-] &= 0 \\
Y_+\vec{y}_- &= Y\vec{y} = Y_-\vec{y}_+ - Y_\pm\vec{y} = (1-Y)\vec{y}_\pm
\end{aligned} \tag{2.39}
$$

To end this paragraph we mention the important fact that in order to use big matrices in practical computation, one has to enlarge the zero mode sector, in order to disentangle the non-anticommuting oscillators (b_0, c_0). This technique is explained in detail in [15].

2.3 The twisted star

In [10] another type of star-product is considered. It represents the gluing condition in a twisted conformal field theory of the ghost system. The twist is done by subtracting to the stress tensor one unit of derivative of the ghost current

$$
T'(z) = T(z) - \partial j_{gh}(z)\,. \tag{2.40}
$$

This redefinition changes the conformal weight of the bc fields from (2,-1) to (1,0). It follows that the background charge is shifted from -3 to -1. As a consequence, in order not to have vanishing correlation functions, we have to fix only one c zero-mode. In particular, the SL$(2,\mathbb{R})$-invariant propagator of the bc system is

$$
\langle b(z)c(w)\rangle' = \frac{1}{z-w}\frac{w-\xi}{z-\xi} \tag{2.41}
$$

where ξ is one fixed point.

In [10] it was shown that the usual product can be obtained from the twisted one by inserting a $n_{gh}=1$-operator at the midpoint which, on singular states like the sliver, can be identified with a c-midpoint insertion. This implies that, on such singular projectors, the twisted product can be identified with the reduced one.

The twisted ghost Neumann coefficients are then defined to be[5]

$$
\begin{aligned}
\tilde{N}'^{ab}_{nm} &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^{m+1}} f'_a(z)\frac{-1}{f_a(z)-f_b(w)}\frac{f_b(w)}{f_a(z)} \\
&= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^{m+1}}\frac{4i}{3}\frac{1}{1+z^2}\frac{\bar{\alpha}^b f(w)}{\bar{\alpha}^a f(z) - \bar{\alpha}^b f(w)}\,.
\end{aligned} \tag{2.42}
$$

As in (2.24) these coefficients refer to the Bra vertex, the corresponding coefficients for the Ket vertex are

$$
\tilde{V}'^{ab}_{nm} = -(-1)^{n+m}\tilde{N}'^{ab}_{nm}\,. \tag{2.43}
$$

We will see in the next section how to compute such coefficients using previous results. This will lead to interesting connections with the other star-products.

[5]We put, for simplicity, $\xi = f_a(i) = 0$

3. Relations among the stars

In this section we will show how the stars products defined above are related to each other. In particular we will show the explicit relations which connect all the Neumann coefficients in the game, so at the end the three star products are homeomorphic and in this sense can be considered equivalent.

3.1 Twisted ghosts vs. matter

The commuting matter Neumann coefficients which appear in (2.18) are given by

$$M_{nm}^{ab} = -\frac{(-1)^m}{\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} f_a'(z) \frac{1}{(f_a(z) - f_b(w))^2} f_b'(w) \,. \tag{3.1}$$

We can rewrite them as

$$\begin{aligned} M_{nm}^{ab} &= -\frac{(-1)^m}{\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} f_a'(z) \partial_w \frac{1}{f_a(z) - f_b(w)} \\ &= -(-1)^m \sqrt{\frac{m}{n}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} \frac{f_a'(z)}{f_a(z) - f_b(w)} \end{aligned} \tag{3.2}$$

where we have integrated by part to respect the variable w. Now, recalling

$$f_a'(z) = \frac{4i}{3} \frac{1}{1+z^2} \alpha^{2-a} f(z) \,, \tag{3.3}$$

we obtain

$$M_{nm}^{ab} = -(-1)^m \sqrt{\frac{m}{n}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} \frac{4i}{3} \frac{1}{1+z^2} \frac{\bar{\alpha}^a f(z)}{\bar{\alpha}^a f(z) - \bar{\alpha}^b f(w)} \,. \tag{3.4}$$

Let us now consider the corresponding twisted ghost Neumann coefficients

$$\begin{aligned} Y_{nm}^{\prime ab} &= (C\tilde{V}^{\prime ab})_{nm} \\ &= (-1)^m \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} \frac{f_a'(z)}{(f_a(z) - f_b(w))} \frac{f_b(w)}{f_a(z)} \\ &= (-1)^m \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} \frac{4i}{3} \frac{1}{1+z^2} \frac{\bar{\alpha}^b f(w)}{(\bar{\alpha}^a f(z) - \bar{\alpha}^b f(w))} \,. \end{aligned} \tag{3.5}$$

This coefficients are not symmetric if we exchange n with m, however we can easily symmetrize them by the use of the matrix $E_{nm} = \sqrt{n}\delta_{nm}$

$$Y^{\prime ab} \to E^{-1} Y^{\prime ab} E \,. \tag{3.6}$$

It is now easy to show the following

$$\begin{aligned} (E^{-1}Y^{\prime ab}E)_{nm} + M_{nm}^{ab} &= (-1)^m \sqrt{\frac{m}{n}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} \frac{4i}{3} \frac{1}{1+z^2} \frac{(\bar{\alpha}^b f(w) - \bar{\alpha}^a f(z))}{(\bar{\alpha}^a f(z) - \bar{\alpha}^b f(w))} \\ &= -(-1)^m \sqrt{\frac{m}{n}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} \frac{4i}{3} \frac{1}{1+z^2} = 0 \end{aligned} \tag{3.7}$$

the last equality holding since there are no poles for $n, m \geq 1$.

So we obtain

$$E^{-1}Y'^{ab}E = -M^{ab} \tag{3.8}$$

a remarkable relation between twisted ghost and matter vertices, which is the same relation that holds in the four-string vertex between the non-twisted ghost and the matter Neumann coefficients [14]. This relation proves also that the ghost integral is independent of the background charge, for $n, m \geq 1$: the matter integral, indeed, can be seen as the ghost integral without the background charge.[6] As a consequence of the relation with the matter coefficients we can derive all the relevant properties of the twisted ghost Neumann coefficients, by simply taking the matter results (2.18) and changing the sign in odd powers.

$$\begin{aligned} Y' + Y'_+ + Y'_- &= -1 \\ Y'^2 + Y'^2_+ + Y'^2_- &= 1 \\ Y'^3_+ + Y'^3_- &= 2Y'^3 + 3Y'^2 - 1 \\ Y'_+Y'_- &= Y'^2 + Y' \\ [Y', Y'_\pm] &= 0 \\ [Y'_+, Y'_-] &= 0\,. \end{aligned} \tag{3.9}$$

3.2 Twisted vs. reduced

The relation between the twisted and non-twisted ghost Neumann coefficients can now be obtained using the previous relation

$$Y' = -EME^{-1} \tag{3.10}$$

and the Gross-Jevicki relation [14][7]

$$Y = E\frac{-M}{1+2M}E^{-1} \tag{3.13}$$

between matter and non-twisted ghosts. So, finally, we have

$$Y = \frac{Y'}{1-2Y'} \tag{3.14}$$

or

$$Y' = \frac{Y}{1+2Y}\,. \tag{3.15}$$

This relation is also strictly related to the equality of solutions between the ghost sliver constructed from the twisted CFT and the non-twisted one [12]. Indeed, it is possible to derive such relation from the equality of ghost algebraic slivers, as we will see in the next section.

[6] The independence of the background charge is also crucial to prove $\tilde{N}'^{ab} = C\tilde{N}'^{ba}C$.

[7] This relation, as noted in [21] contains the map

$$P(z) = \frac{-z}{1+2z} \tag{3.11}$$

which is a $\mathrm{PSL}(2,\mathbb{R})$ transformation that squares to itself

$$P \circ P(z) = z\,. \tag{3.12}$$

4. Slivers

In this section we review the algebraic derivation of the sliver state in matter and ghost sector. Then we compute algebraically the slivers in the twisted ghost sector and show how identity of such states is implied by the relations (3.14) between the Neumann coefficients in the game.

4.1 Matter sliver and ghost solution

The projection equation in the matter sector

$$|\psi\rangle_m = |\psi\rangle_m *_m |\psi\rangle_m \tag{4.1}$$

can be solved as in [7, 9], by the ansatz

$$\begin{aligned} |\psi\rangle_m &= \mathcal{N}_m \exp\left(\sum_{n,m\geq 1} a_n^\dagger S_{nm} a_m^\dagger\right)|0\rangle \\ S &= CSC \end{aligned} \tag{4.2}$$

where

$$T = CS = \frac{1}{2M}\left(1 + M - \sqrt{(1-M)(1+3M)}\right). \tag{4.3}$$

The ghost equation of motion is

$$\mathcal{Q}|\psi\rangle_g + |\psi\rangle_g *_g |\psi\rangle_g = 0\,. \tag{4.4}$$

This equation is easy to solve if we use big matrices in order to handle at the same time both zero and non zero modes (see [15]). The relevant results are

$$|\psi\rangle_g = \tilde{\mathcal{N}}_g \exp\left(\sum_{n,m\geq 1} c_n^\dagger \tilde{S}_{nm} b_m^\dagger\right)|\dot{0}\rangle \tag{4.5}$$

$$\tilde{T} = C\tilde{S} = \frac{1}{2Y}\left(1 + Y - \sqrt{(1-Y)(1+3Y)}\right) \tag{4.6}$$

$$\mathcal{Q} = c_0 + \vec{f}\cdot(\vec{c} + C\vec{c}^\dagger) \tag{4.7}$$

$$\vec{f} = \frac{\vec{y}}{1-Y}\,. \tag{4.8}$$

Using the integral representations (2.28) one can actually prove that $\mathcal{Q}$ is a midpoint insertion [15, 13]

$$\mathcal{Q} = c_0 + \sum_{n=1}^{\infty}(-1)^n(c_{2n} + c_{-2n}) = \frac{1}{2i}\left(c(i) - c(-i)\right). \tag{4.9}$$

4.2 The twisted sliver in the algebraic approach

We have seen that the Neumann coefficients of the star product in the twisted CFT coincides to (minus) the matter ones at zero momentum. This implies that we can solve the algebraic equation for projectors, as for the usual ghost star product, but now using the linear and non linear relations (3.9).

So we impose the projector equation

$$|S\rangle' = |S\rangle' *' |S\rangle' \tag{4.10}$$

with the ansatz

$$|S\rangle' = \mathcal{N} \exp\left(\sum_{n,m\geq 1} c_n^\dagger S'_{nm} b_m^\dagger\right)|0'\rangle \tag{4.11}$$

we can safely follow the non twisted case [8, 15] and arrive at the equation

$$T' = CS' = Y' + (Y'_+, Y'_-)\frac{1}{1-\Sigma'\mathcal{V}'}\Sigma'\begin{pmatrix} Y'_- \\ Y'_+ \end{pmatrix} \tag{4.12}$$

where

$$\Sigma' = \begin{pmatrix} T' & 0 \\ 0 & T' \end{pmatrix}, \qquad \mathcal{V}' = \begin{pmatrix} Y' & Y'_+ \\ Y'_- & Y' \end{pmatrix}.$$

using the properties (5.37) we obtain the algebraic equation

$$(T'+1)(-Y'T'^2 + (1-Y')T' - Y') = 0 \tag{4.13}$$

which, apart from the trivial solution $T' = -1$, gives[8]

$$T' = CS' = \frac{1}{2Y'}\left(1 - Y' - \sqrt{(1+Y')(1-3Y')}\right). \tag{4.14}$$

It is interesting to compare it with the algebraic projector w.r.t. the reduced product

$$\tilde{T} = \frac{1}{2Y}\left(1 + Y - \sqrt{(1-Y)(1+3Y)}\right). \tag{4.15}$$

The equality of this two solutions holds if and only if the following relation between twisted and non twisted Neumann coefficients is obeyed

$$Y' = \frac{Y}{1+2Y} \tag{4.16}$$

which is exactly (3.14). This shows that equality of solution in VSFT is equivalent to the statement (3.13) which, on the other hand, have its explanation via the 4-string vertex [14]. It would be interesting then to explore further the relations (if any) between matter and ghost in the 4-string vertex and matter and twisted ghost in the conformal-three-string vertex.

[8] As usual we choose the square root branch cut which doesn't have divergence as $Y' \to 0$.

5. Diagonalization

In this section we compute the spectrum of the twisted product and verify that is related to the matter one by (3.10). Then we do the same thing on the usual ghost product. In this case the diagonalization of "big" matrices can be carried over in two step. First we block diagonalize them isolating completely the (0,0) component: this will determine a sort of discrete spectrum. Second we diagonalize the internal (small) matrices, using the results from the twisted CFT, in order to find the usual continuous spectrum.[9]

5.1 Diagonalization of the twisted product

Knowing the fact that twisted Neumann coefficients can be easily symmetrized to take the form of (minus) the matter Neumann coefficients, we have for free the eigenvalues. However we would like to show, as a consistency check, that we can derive the twisted spectrum by purely conformal considerations, following the lines of [18] but now using the twisted conformal field theory of the ghost system. As we have seen before, the twist is done as

$$T'(z) = T(z) - \partial j_{gh}(z) \tag{5.1}$$

leading to

$$L'_n = L_n + n j_n + \delta_{n0} \tag{5.2}$$

where

$$\begin{aligned} L_n &= -\sum_{k=-\infty}^{\infty} (2n-k) : c_{n-k} b_k : \\ j_n &= \sum_{k=-\infty}^{\infty} : c_{n-k} b_k : \ . \end{aligned} \tag{5.3}$$

To find the eigenvectors of Y we consider the $*'$ algebra derivation

$$K'_1 = L'_1 + L'_{-1} \tag{5.4}$$

and then we use the same formal arguments of [18]. The main difference here is that K'_1 acts on b and c oscillator in a different but complementary way, due to their (twisted) conformal properties

$$[K'_1, c_n] = -(n+1)c_{n+1} - (n-1)c_{n-1} \tag{5.5}$$

$$[K'_1, b_n] = -n\, b_{n+1} - n\, b_{n-1}\, . \tag{5.6}$$

We can have c-type vectors v_n, as well as b-type vectors w_n, so K_1 has two different matrix representations. If we act on c oscillators we get

$$\begin{aligned} [K'_1, v_n c_n] &= (K^{(c)} v) \cdot c \\ K^{(c)}_{nm} &= -(m+1)\delta_{n,m+1} - (m-1)\delta_{n,m-1}\, . \end{aligned} \tag{5.7}$$

[9] The topic of ghost spectroscopy in Siegel gauge was also treated in [22, 23, 24].

If we act on b oscillators we get

$$\begin{aligned}[K_1', w_n b_n] &= (K^{(b)} v) \cdot b - w_1 b_0 \\ K^{(b)}_{nm} &= -m\delta_{n,m+1} - m\delta_{n,m-1}\,. \end{aligned} \tag{5.8}$$

These two matrices transpose to each other and obey

$$K^{(c)} = K^{(b)\,T} = A^{-1} K^{(b)} A \tag{5.9}$$

in particular they share eigenvalues. The matrix A is defined to be

$$A_{nm} = n\delta_{nm}\,. \tag{5.10}$$

We shall begin by diagonalizing $K^{(c)}$ and determine its eigenvectors.

$$K^{(c)}\, v^{\kappa} = \kappa\, v^{\kappa}\,. \tag{5.11}$$

In order to do so we map this algebraic problem in a differential one, by defining the generating function

$$f_{v^\kappa}(z) = \sum_{n=1}^{\infty} v_n^\kappa z^n \tag{5.12}$$

so that

$$v_n^\kappa = \oint_0 \frac{dz}{2\pi i} \frac{1}{z^{n+1}} f_{v^\kappa}(z)\,. \tag{5.13}$$

With trivial manipulations we find that (5.11) is equivalent to

$$\left(-(1+z^2)\frac{d}{dz} - (z - \frac{1}{z}) \right) f_{v^\kappa}(z) = \kappa\, f_{v^\kappa}(z) \tag{5.14}$$

which is easily integrated to give

$$f_{v^\kappa}(z) = \frac{z}{z^2+1} \mathrm{e}^{-\kappa \tan^{-1} z} \tag{5.15}$$

where we have chosen the overall normalization in order to $v_1^\kappa = 1$. As usual κ is a continuous parameter spanning all the real axis.

To find the b-eigenvectors it is worth noting that $K^{(b)}$ is the same as in the matter case [18], so we simply get the result

$$f_{w^\kappa}(z) = \frac{1}{k}(1 - \mathrm{e}^{-\kappa \tan^{-1} z})\,. \tag{5.16}$$

As a consistency check note that due to (5.9) c-eigenvectors are related to b-eigenvectors by

$$v_n^\kappa = n\, w_n^\kappa \tag{5.17}$$

which in functional language reads

$$f_{v^\kappa}(z) = z \frac{d}{dz} f_{w^\kappa}(z)\,. \tag{5.18}$$

It is trivial to check that this relation is identically satisfied.

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Once the spectrum of K'_1 is found, in order to find the spectrum of Y, we begin by considering the algebra of wedge states in the twisted CFT. A wedge state can be defined as

$$|N\rangle' = \left(|0\rangle'\right)_{*'}^{N-1} = \mathcal{N}'_N \exp\left(\sum_{n,m=1}^{\infty} c_n^\dagger\,(CT'_N)_{nm}\, b_m^\dagger\right)|0\rangle'\,. \tag{5.19}$$

These states satisfy the relation

$$|N+1\rangle' = |N\rangle' *' |0\rangle'\,. \tag{5.20}$$

Following the same formal arguments of [25], we can write all T_N in function of the sliver matrix T[10]

$$T'_N = \frac{T' - T'^{N-1}}{1 - T'^N}\,. \tag{5.21}$$

In particular we have

$$T'_2 = 0 \tag{5.22}$$
$$T'_3 = Y' \tag{5.23}$$
$$T'_\infty = T' \tag{5.24}$$

actually the last equation is well defined for $|T| \leq 1$, we will see a posteriori that the eigenvalues of T lie on the interval $(0,1]$.

Such wedge states can be defined as surface states in the twisted CFT [10]. Given a string field $|\phi\rangle = \phi'(0)|0\rangle'$ the wedge state $|N\rangle$ can be defined as[11]

$$'\langle N|\phi\rangle = \langle f_N \circ \phi(0)\rangle' \tag{5.25}$$

where the generating function of the surface state is given by

$$f_N(z) = \frac{N}{2}\tan\left(\frac{2}{N}\tan^{-1} z\right) \tag{5.26}$$

Now we consider the state $|2+\epsilon\rangle'$. This state can be given a representation in terms of the twisted Virasoro generators as [26]

$$|B\rangle' = \exp\left(\epsilon V_-\right)|0\rangle' = |0\rangle' + \epsilon V'_-|0\rangle' + O(\epsilon^2) \tag{5.27}$$
$$V_- = \sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n-1)(2n+1)} L'_{-2n} \tag{5.28}$$

Using the explicit form of the twisted Virasoro generators

$$L'_n = -\sum_{k=-\infty}^{\infty}(n-k) : c_{n-k} b_k : \tag{5.29}$$

[10] Note the change of signs with respect to [25], they come out from the differences in the algebraic linear and non linear properties of the Neumann coefficients of the twisted CFT

[11] In the brackets insertion of the c_0 0 mode is intended, since all oscillators in the game start from the 1 component, we don't have any ambiguity.

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we can find the relevant Neumann coefficients of the state $|2+\epsilon\rangle'$

$$|2+\epsilon\rangle' = \exp\left(\epsilon \sum_{n,m=1}^{\infty} c_n^\dagger \,(CB')_{nm}\, b_m^\dagger\right)|0\rangle' = |0\rangle' + \epsilon \sum_{n,m=1}^{\infty} c_n^\dagger \,(CB')_{nm}\, b_m^\dagger |0\rangle' + O(\epsilon^2) \tag{5.30}$$

the B'_{nm} coefficients can be computed by comparing (5.30) with (5.28), we get

$$B'_{nm} = \frac{1}{2}\left(1+(-1)^{n+m}\right)\frac{(-1)^{\frac{n-m}{2}} n}{(n+m)^2-1}\,. \tag{5.31}$$

This coefficient is made diagonal with c-type eigenvectors

$$\sum_{m=1}^{\infty} B'_{nm} v_m^\kappa = \sum_{m=1}^{\infty} B'_{nm} m\, w_m^\kappa = \beta'(\kappa) v_n^\kappa = \beta'(\kappa) n\; w_n^\kappa\,, \tag{5.32}$$

Now take $n=1$, all goes the same way as [18], except for a minus sign in the definition (5.31)

$$\beta'(\kappa) = \frac{1}{2}\,\frac{\frac{\pi\kappa}{2}}{\sinh\frac{\pi\kappa}{2}}\,. \tag{5.33}$$

From B'-eigenvalues we can find out the eigenvalues of the twisted sliver $\tau'(k)$, by inverting the relation (5.21) at $N = 2+\epsilon$

$$B' = \frac{T'\log(T')}{1-T'} \tag{5.34}$$

which is bijective in the range $T \in (0,1]$, in so doing we get

$$\tau'(k) = \mathrm{e}^{-\frac{\pi|\kappa|}{2}}\,. \tag{5.35}$$

Then we can use the twisted wedge states formula at $N = 3$ to get the eigenvalues of Y', which we call $y'(\kappa)$

$$y'(\kappa) = \frac{\tau'(k) - \tau'(k)^2}{1-\tau'(k)^3} = \frac{1}{2\cosh\frac{\pi\kappa}{2}+1}\,. \tag{5.36}$$

To find the spectrum of the other two coefficients $Y'_\pm$ we use the relations

$$\begin{aligned} Y' + Y'_+ + Y'_- &= -1 \\ Y'_+ Y'_- &= Y'^2 + Y' \end{aligned}$$

solving them for $Y'_\pm$ we get

$$Y'_\pm = -\frac{1}{2}\left(1+Y' \mp \sqrt{(1-3Y')(1+Y')}\right) = -\frac{1+\cosh\frac{\pi\kappa}{2} \pm \sinh\frac{\pi\kappa}{2}}{2\cosh\frac{\pi\kappa}{2}+1}\,. \tag{5.37}$$

As expected they are exactly the opposite of the matter ones.

5.2 Block diagonalization of non twisted star

Let's rewrite for the sake of clarity the general form of the matrices defining the usual ghost product

$$\mathcal{Y} = \begin{pmatrix} 1 & 0 \\ \vec{y} & Y \end{pmatrix} \tag{5.38}$$

$$\mathcal{Y}_\pm = \begin{pmatrix} 0 & 0 \\ \vec{y}_\pm & Y_\pm \end{pmatrix}. \tag{5.39}$$

The $(0,0)$ component isolates one eigenvalue for each matrix

$$eig[\mathcal{Y}] = 1 \oplus eig[Y] \tag{5.40}$$

$$eig[\mathcal{Y}_\pm] = 0 \oplus eig[Y_\pm]. \tag{5.41}$$

It is then straightforward to find the eigenvector relative to these eigenvalues, this is achieved by block diagonalizing such matrices

$$\hat{\mathcal{Y}} = \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix} \tag{5.42}$$

$$\hat{\mathcal{Y}}_\pm = \begin{pmatrix} 0 & 0 \\ 0 & Y_\pm \end{pmatrix} \tag{5.43}$$

with the change of basis

$$\hat{\mathcal{Y}}_{(\pm)} = \mathcal{Z}^{-1}\mathcal{Y}_{(\pm)}\mathcal{Z} \tag{5.44}$$

$$\mathcal{Z} = \begin{pmatrix} 1 & 0 \\ \vec{f} & 1 \end{pmatrix} \tag{5.45}$$

$$\mathcal{Z}^{-1} = \begin{pmatrix} 1 & 0 \\ -\vec{f} & 1 \end{pmatrix} \tag{5.46}$$

where

$$\vec{f} = \frac{1}{1-Y}\vec{y} = -\frac{1}{Y_\pm}\vec{y}_\pm\,. \tag{5.47}$$

The equality of the last expressions is a simple consequence of (2.39). Note that the eigenvector we find is the same which defines the kinetic operator Q as a c midpoint insertion.[12] Since the equation (5.47) has the solution (4.9), it might seem that $(Y, Y_\pm)$ (small matrices) cannot have the eigenvalues $(1,0)$, this is actually not true because (5.47) is not a relation in the full Hilbert space, but only in its twist-even subspace. As we will see the $(1,0)$ eigenvalues will have a corresponding one twist odd eigenvector, contrary with all the other eigenvalues which will have eigenvectors of both twist parity. The linear transformation (5.45) induces the following redefinition of the bc oscillators.

$$\tilde{c}_0 = c_0 + \sum_{n\geq 1} f_n(c_n + (-1)^n c_n^\dagger) = \mathcal{Q} \tag{5.48}$$

$$\tilde{c}_n = c_n \quad n \neq 0 \tag{5.49}$$

$$\tilde{b}_0 = b_0 \tag{5.50}$$

$$\tilde{b}_n = -f_n b_0 + b_n \quad n \neq 0 \tag{5.51}$$

[12] This, from a different point of view, was also note in [27].

where we have defined ($f_{-n} \equiv f_n$). This is an equivalent representation of the bc system[13]

$$\{\tilde{b}_N, \tilde{c}_M\} = \delta_{N+M} \quad N, M = -\infty, \ldots, 0, \ldots, \infty\,. \tag{5.52}$$

Block diagonalization of big matrices has then lead to the discovery of a twist even eigenvector with non vanishing 0-component. This eigenvector is not visible in Siegel gauge and, as we have seen, it corresponds to the midpoint of the (ghost part of the) string.

5.3 Diagonalization of the reduced product

Once we know the spectrum of the twisted product we can use the equality of the twisted sliver and the reduced sliver (i.e. sliver in Siegel gauge) to directly compute the spectrum of the reduced Neumann coefficients. Here again we can define "wedge"-states as

$$|N\rangle = \left(|\dot{0}\rangle\right)^{N-1}_{*_{b_0}} = \mathcal{N}_N \exp\left(\sum_{n,m=1}^{\infty} c_n^\dagger\,(CT_N)_{nm}\, b_m^\dagger\right)|\dot{0}\rangle\,. \tag{5.53}$$

Which are defined by

$$|N+1\rangle = |N\rangle *_{b_0} |\dot{0}\rangle\,. \tag{5.54}$$

The Neumann coefficients T_N, are given by[14]

$$T_N = \frac{T + (-T)^{N-1}}{1-(-T)^N}\,. \tag{5.55}$$

In particular we have

$$T_2 = 0 \tag{5.56}$$

$$T_3 = Y \tag{5.57}$$

$$T_\infty = T = T'\,. \tag{5.58}$$

The last equality follows from the fact that the twisted sliver is identical to the reduced sliver in Siegel gauge. We recall here that the name "wedge states" is somehow misleading, since this states cannot be interpreted as surface state in the non twisted CFT, this is so because the star product in the usual CFT increase the ghost number (as opposed to the twisted star product). In this sense these states can be properly defined only algebraically via the reduced product.

At $N = 3$, we get the eigenvalues of Y, $y(\kappa)$, from the eigenvalues of $T = T'$ (5.35)

$$y(\kappa) = \frac{1}{2\cosh\frac{\pi\kappa}{2} - 1}\,. \tag{5.59}$$

Using (2.39) we obtain the spectrum for the Neumann coefficients $Y_\pm$, which we call $y_\pm(\kappa)$

$$y_\pm(\kappa) = \frac{\cosh\frac{\pi\kappa}{2} \pm \sinh\frac{\pi\kappa}{2} - 1}{2\cosh\frac{\pi\kappa}{2} - 1}\,. \tag{5.60}$$

[13]In order to prove this, twist invariance of $\vec{f}$ is crucial ($C\vec{f} = \vec{f}$)

[14]The expression is formally identical to the matter case, this is so because the linear and non linear properties of the reduced Neumann coefficients are isomorphic to the matter.

6. Conclusions

In this paper we have shown how the zero momentum sector of the matter and the Siegel gauge part of the ghosts are related by a twist in the CFT of the bc system.

The fact that the ghost vertex is related to the matter sector is not a novelty and is known since [14], however this correspondence was shown there indirectly via the 4-strings vertex.

We have indeed proven that this correspondence can be understood by the twist of the bc system and the equality of the twisted sliver and the sliver in Siegel gauge. We have, in particular, showed that the twisted CFT defines exactly the matter Neumann coefficients up to a minus sign. The "physical" meaning of this deep relation (which connects two very different CFT's, one of second order and the other of first order) is probably related to the fact that the ghosts eliminate the unphysical degrees of freedom of the string, so it can be expected that there exists a representation of them in which they are isomorphic to the non dynamical DF of the matter. This representation is the twist where the central charge is -2, the opposite of two light-cone matter directions.

The slivers of the three sectors are the same (up to signs and symmetrizing factors), even though they are defined from star products which are manifestly different. This can possibly be traced back to the fact that these projectors exhibit a singular geometry [28] and, in this sense, are insensitive of the underlying CFT. It would be interesting to see if such equality still holds for more general and less singular projectors like the butterflies and others studied in [11].

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